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An alternative derivation of closed-form representations for multidimensional lattice sums of generalized hypergeometric functions

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Abstract
Some representations for infinite m-dimensional lattice sums of generalized hypergeometric functions, which were deduced previously by the first-named author, are rederived here by appealing to an alternative more direct method. In particular, it is shown that these lattice sums are proportional to a finite sum of Meijer's G-functions.

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1. Introduction

Let $\boldsymbol{q}\,(m)$ denote the vector whose m components $(m \geqslant 1)$ range over the set $\mathbb{Z}$ of all integers (positive, negative, and zero). The length of an arbitrary vector $\boldsymbol{\alpha}\,(m)$ is denoted by $\alpha\,(m)$ so that, for example, if $q_j \in \mathbb{Z}$ and

$$\boldsymbol{q}\,(m) = (q_1, q_2, \ldots, q_m)$$

then

$$q\,(m) = \left(q_1^2 + q_2^2 + \cdots + q_m^2\right)^{1/2}.$$

Moreover, for real α_j, if

$$\boldsymbol{\alpha}\,(m) = (\alpha_1, \alpha_2, \ldots, \alpha_m)$$

then

$$\boldsymbol{\alpha}\,(m) \cdot \boldsymbol{q}\,(m) = \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_m q_m$$

denotes the dot product of the vectors $\boldsymbol{\alpha}\,(m)$ and $\boldsymbol{q}\,(m)$. When $m = 1$, an arbitrary vector $\boldsymbol{\alpha}\,(1)$ is considered a scalar and the dot product of two such vectors denotes ordinary multiplication of scalars.

Recently, Miller [4] has shown by appealing to the principle of mathematical induction that the m-dimensional lattice sum:

$$\sum_{q(m)} \cos(2\pi \boldsymbol{\alpha}\cdot\boldsymbol{q})\ {}_pF_{p+1}\left[(a_p);(b_{p+1});-x^2q^2\right] = \frac{2\pi^{-m/2}}{\Gamma(m/2)} \sum_{q(m)}^{\omega^2\leqslant x^2/\pi^2} \int_0^\infty t^{m-1} \times {}_0F_1\left[\begin{matrix} \text{——};\\ m/2;\end{matrix}\ -\omega^2t^2\right] {}_pF_{p+1}\left[\begin{matrix}(a_p);\\(b_{p+1});\end{matrix}\ -\frac{x^2t^2}{\pi^2}\right]\mathrm{d}t \tag{1.1}$$

where $x > 0$, $m \geqslant 1$, and $\omega(m) = |\boldsymbol{\alpha}(m) + \boldsymbol{q}(m)|$. The integral in equation (1.1) converges when $\omega^2 < x^2/\pi^2$ provided that

$$\Re(a_k) > \tfrac{1}{4}(m-1) \qquad \text{and} \qquad \Re(\Delta) > \tfrac{1}{2}m \tag{1.2a}$$

and when $\omega^2 \leqslant x^2/\pi^2$ provided that

$$\Re(a_k) > \tfrac{1}{4}(m-1) \qquad \text{and} \qquad \Re(\Delta) > \tfrac{1}{2}m+1 \tag{1.2b}$$

where $1 \leqq k \leqq p$ and Δ is defined by

$$\Delta := \sum_{j=1}^{p+1} b_j - \sum_{j=1}^{p} a_j.$$

When $p = 0$, the latter conditional inequalities for $\Re(a_k)$ become superfluous and $\Delta = b_1$. Furthermore, the inequalities (1.2) evidently provide necessary conditions for the convergence of the m-dimensional lattice sum in equation (1.1); for necessary and sufficient conditions see [4, lemma 1]. We mention, however, that the lattice sums under consideration converge absolutely provided that $\Re(a_k) > \frac{1}{2}m$ and $\Re(\Delta) > m + \frac{1}{2}$. Moreover, the integral in equation (1.1) may be evaluated by using [6, equations (4.4) and (5.1)] and written essentially in terms of $p + 1$ generalized hypergeometric functions ${}_{p+1}F_p\left[\pi^2\omega^2/x^2\right]$. The result just alluded to is given by equation (3.11) below. (See, for example, [8, p 19] for an introduction to generalized hypergeometric functions.)

The derivation of equation (1.1) given in section 3 relies upon the following result which we shall prove inductively in section 2.

Lemma 1. *For integers $m \geqslant 1$*

$$\int_0^1\int_0^1\cdots\int_0^1 x_m^m\, x_{m-1}^{m-1}\cdots x_1\, J_0\left(r\sqrt{\beta_{-1}^2+\beta_0^2}\,x_1x_2\cdots x_m\right)\frac{\cos\left(r\beta_m\sqrt{1-x_m^2}\right)}{\sqrt{1-x_m^2}} \times\frac{\cos\left(r\beta_{m-1}x_m\sqrt{1-x_{m-1}^2}\right)}{\sqrt{1-x_{m-1}^2}}\cdots\frac{\cos\left(r\beta_1x_2\cdots x_m\sqrt{1-x_1^2}\right)}{\sqrt{1-x_1^2}}\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_m = \left(\frac{\pi}{2r\beta}\right)^{m/2} J_{m/2}(r\beta) \tag{1.3}$$

where the β_j $(-1 \leqslant \beta_j \leqslant m)$ and r are complex numbers, and

$$\beta = \left(\beta_{-1}^2+\beta_0^2+\beta_1^2+\cdots+\beta_m^2\right)^{1/2}.$$

Note that, when $m = 0$, equation (1.3) reduces trivially to an identity, since there are no integrations.

2. Inductive proof of equation (1.3)

When $m = 1$, equation (1.3) reduces to the well known result (cf., e.g., [2, p. 425] and [7, section 2.12.21, equation (6)] or in equation (2.4) below set $\nu = 0$, $b = r\beta_1$, and $c = r\sqrt{\beta_{-1}^2 + \beta_0^2}$):

$$\int_0^1 \frac{x_1 \cos\left(r\beta_1\sqrt{1-x_1^2}\right)}{\sqrt{1-x_1^2}} J_0\left(rx_1\sqrt{\beta_{-1}^2+\beta_0^2}\right) \mathrm{d}x_1 = \left(\frac{\pi}{2r\beta}\right)^{\frac{1}{2}} J_{\frac{1}{2}}(r\beta) \tag{2.1}$$

where

$$\beta = \left(\beta_{-1}^2 + \beta_0^2 + \beta_1^2\right)^{1/2}.$$

Now, denoting the left-hand side of equation (1.3) by $S(m)$, we intend to evaluate the following multiple integral:

$$\begin{aligned} S(m+1) = \int_0^1 x_{m+1}^{m+1} \frac{\cos\left(r\beta_{m+1}\sqrt{1-x_{m+1}^2}\right)}{\sqrt{1-x_{m+1}^2}} \Bigg[\int_0^1 \cdots \int_0^1 x_m^m \cdots x_1 \\ \times J_0\left(rx_{m+1}\sqrt{\beta_{-1}^2+\beta_0^2}\, x_1\cdots x_m\right) \frac{\cos\left(rx_{m+1}\beta_m\sqrt{1-x_m^2}\right)}{\sqrt{1-x_m^2}} \cdots \\ \times \frac{\cos\left(rx_{m+1}\beta_1 x_2\cdots x_m\sqrt{1-x_1^2}\right)}{\sqrt{1-x_1^2}} \mathrm{d}x_1\cdots \mathrm{d}x_m\Bigg] \mathrm{d}x_{m+1}. \end{aligned} \tag{2.2}$$

Assuming that equation (1.3) is true for an arbitrary positive integer m (this is the induction hypothesis), the integrations with respect to $x_1, \ldots, x_m$ yield

$$\left(\frac{\pi}{2rx_{m+1}\alpha}\right)^{m/2} J_{m/2}(rx_{m+1}\alpha)$$

where

$$\alpha = \left(\beta_{-1}^2 + \beta_0^2 + \beta_1^2 + \cdots + \beta_m^2\right)^{1/2}.$$

Thus equation (2.2) gives

$$S(m+1) = \left(\frac{\pi}{2r\alpha}\right)^{m/2} \int_0^1 x_{m+1}^{m/2+1} J_{m/2}(r\alpha x_{m+1}) \frac{\cos\left(r\beta_{m+1}\sqrt{1-x_{m+1}^2}\right)}{\sqrt{1-x_{m+1}^2}} \mathrm{d}x_{m+1}. \tag{2.3}$$

The latter integral is a specialization of

$$\int_0^1 t^{\nu+1} J_\nu(ct) \frac{\cos\left(b\sqrt{1-t^2}\right)}{\sqrt{1-t^2}} \mathrm{d}t = \sqrt{\frac{\pi}{2}} c^\nu \left(b^2+c^2\right)^{-\frac{2\nu+1}{4}} J_{\nu+\frac{1}{2}}\left(\sqrt{b^2+c^2}\right) \tag{2.4}$$

where $\Re(\nu) > -1$. Although this result is well known (cf., e.g., [7, section 2.12.21, equation (5)]), an easy derivation is alluded to in the discussion pertaining to [4, equation (4.1)] which is essentially the same result as equation (2.4). Now, setting $\nu = m/2$, $c = r\alpha$, $b = r\beta_{m+1}$ in equation (2.4), we find from equation (2.3) that

$$S(m+1) = \left(\frac{\pi}{2r\beta}\right)^{\frac{m+1}{2}} J_{\frac{m+1}{2}}(r\beta) \tag{2.5}$$

where

$$\beta = \left(\beta_{-1}^2 + \beta_0^2 + \beta_1^2 + \cdots \beta_{m+1}^2\right)^{1/2}$$

which is just equation (1.3) with m replaced by $m + 1$. This evidently completes the proof of equation (1.3) by induction.

3. Derivation of equation (1.1)

Although the derivation of equation (1.1) that follows is more straightforward, conceptually simpler, and relies on a smaller number of formal manipulations than the (explicitly) inductive derivation given previously in [4], it is nevertheless implicitly inductive in nature, since it depends on the (inductively proved) result given by equation (1.3). Moreover, essentially as the two-dimensional result (i.e., [5, theorem 2]) relies on a polar coordinate transformation and the three-dimensional result (i.e., [4, equation (3.8)]) relies on a spherical coordinate transformation, so too the alternative derivation for the m-dimensional case will require the polar coordinate transformation in m dimensions (see, e.g., [1, equation (9.4.1)]) given by

$$\begin{aligned} x_1 &= r\cos\varphi_1 \\ x_2 &= r\sin\varphi_1\cos\varphi_2 \\ x_3 &= r\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ &\vdots \\ x_{m-1} &= r\sin\varphi_1\cdots\sin\varphi_{m-2}\cos\theta \\ x_m &= r\sin\varphi_1\cdots\sin\varphi_{m-2}\sin\theta \end{aligned} \tag{3.1a}$$

where $m \geqslant 2$, $0 \leqslant \varphi_i \leqslant \pi$, $0 \leqslant \theta \leqslant 2\pi$, and the Jacobian J of the transformation [1, p 456] is given by

$$J(r, \varphi_1, \ldots, \varphi_{m-2}) = r^{m-1}\sin^{m-2}\varphi_1\cdots\sin^2\varphi_{m-3}\sin\varphi_{m-2}. \tag{3.1b}$$

When $m = 2$, the product of sines in equations (3.1) are empty, and so $J(r) = r$.

We employ a form of the m-dimensional Poisson summation formula. Therefore, we shall have to evaluate the m-dimensional Fourier transform $\mathcal{F}$ of the generalized hypergeometric function ${}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2x^2\right]$, where $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ and $t > 0$. Thus, for real ξ_j letting $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_m)$, we find upon using equations (3.1) that

$$\begin{aligned} &\mathcal{F}\left\{{}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2x^2\right]\right\} \\ &\quad= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}\,{}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2x^2\right]\mathrm{d}x_1\cdots\mathrm{d}x_m \\ &\quad= \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{\infty} \mathrm{e}^{\mathrm{i}r(\xi_1\cos\varphi_1+\xi_2\sin\varphi_1\cos\varphi_2+\cdots+\xi_{m-2}\sin\varphi_1\cdots\sin\varphi_{m-3}\cos\varphi_{m-2})} \\ &\qquad\times\mathrm{e}^{\mathrm{i}r\sin\varphi_1\cdots\sin\varphi_{m-2}(\xi_{m-1}\cos\theta+\xi_m\sin\theta)}r^{m-1}\,{}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2r^2\right] \\ &\qquad\times\sin^{m-2}\varphi_1\cdots\sin\varphi_{m-2}\mathrm{d}r\,\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{m-2}\mathrm{d}\theta. \end{aligned} \tag{3.2}$$

However, since

$$J_0\left(r\sqrt{u^2+v^2}\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}r(u\cos\theta+v\sin\theta)}\mathrm{d}\theta$$

clearly we have

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}r\sin\varphi_1\cdots\sin\varphi_{m-2}(\xi_{m-1}\cos\theta+\xi_m\sin\theta)}\mathrm{d}\theta = 2\pi J_0\left(r\sin\varphi_1\cdots\sin\varphi_{m-2}\sqrt{\xi_{m-1}^2+\xi_m^2}\right).$$

And so we find from equation (3.2) that

$$\begin{aligned}\mathcal{F}\left\{{}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2x^2\right]\right\}\\ = 2^{m-1}\pi\int_0^\infty r^{m-1}I(\boldsymbol{\xi};r)\ {}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2r^2\right]\mathrm{d}r \qquad (3.3)\end{aligned}$$

where

$$\begin{aligned}I(\boldsymbol{\xi};r) := \int_0^{\pi/2}\cdots\int_0^{\pi/2}\sin^{m-2}\varphi_1\cdots\sin\varphi_{m-2}J_0\left(r\sin\varphi_1\cdots\sin\varphi_{m-2}\sqrt{\xi_{m-1}^2+\xi_m^2}\right)\\ \times\cos(r\xi_1\cos\varphi_1)\cos(r\xi_2\sin\varphi_1\cos\varphi_2)\cdots\\ \times\cos(r\xi_{m-2}\sin\varphi_1\cdots\sin\varphi_{m-3}\cos\varphi_{m-2})\,\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{m-2}. \qquad (3.4)\end{aligned}$$

Note that, when $m=2$, there are no integrations with respect to $\varphi_1,\ldots,\varphi_{m-2}$, and so it is evident that (in this case we compare equation (3.3) with [5, equation (3.6*a*)])

$$I(\boldsymbol{\xi}(2);r) := J_0\left(r\sqrt{\xi_1^2+\xi_2^2}\right).$$

In the multiple integral $I(\boldsymbol{\xi};r)$, we make use of the transformation:

$$\begin{aligned}\sin\varphi_1 &= t_{m-2}\\ \sin\varphi_2 &= t_{m-3}\\ &\vdots\\ \sin\varphi_{m-2} &= t_1.\end{aligned}$$

Thus equation (3.4) becomes

$$\begin{aligned}I(\boldsymbol{\xi};r) = \int_0^1\cdots\int_0^1 t_{m-2}^{m-2}\cdots t_1\,J_0\left(rt_1\cdots t_{m-2}\sqrt{\xi_{m-1}^2+\xi_m^2}\right)\frac{\cos\left(r\xi_1\sqrt{1-t_{m-2}^2}\right)}{\sqrt{1-t_{m-2}^2}}\\ \times\frac{\cos\left(r\xi_2t_{m-2}\sqrt{1-t_{m-3}^2}\right)}{\sqrt{1-t_{m-3}^2}}\cdots\frac{\cos\left(r\xi_{m-2}t_2\cdots t_{m-2}\sqrt{1-t_1^2}\right)}{\sqrt{1-t_1^2}}\,\mathrm{d}t_1\cdots\mathrm{d}t_{m-2}\end{aligned}$$

which, upon appealing to lemma 1 with m replaced by $m-2$, gives

$$I(\boldsymbol{\xi};r) = \left(\frac{\pi}{2r\xi}\right)^{\frac{m-2}{2}}J_{\frac{m-2}{2}}(r\xi)$$

where

$$\xi = \left(\xi_1^2+\xi_2^2+\cdots+\xi_m^2\right)^{1/2}.$$

Hence we may write equation (3.3) as follows:

$$\mathcal{F}\left\{ {}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2x^2\right]\right\}$$
$$= \frac{(2\pi)^{m/2}}{\xi^{\frac{m-2}{2}}}\int_0^\infty r^{\frac{m}{2}} J_{\frac{m-2}{2}}(\xi r)\; {}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2r^2\right]\mathrm{d}r$$
$$= \frac{2\pi^{m/2}}{\Gamma(m/2)}\int_0^\infty r^{m-1}\,{}_0F_1\left[\begin{matrix} \text{———}; \\ m/2; \end{matrix}\; -\frac{\xi^2r^2}{4}\right] {}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2r^2\right]\mathrm{d}r. \tag{3.5}$$

Either integral in equation (3.5) converges when $\xi^2 \neq 4t^2$ provided that the inequalities (1.2a) hold true (see [6, equations (3.4)]). Moreover, the penultimate integral is a specialization of a generalization of the discontinuous integral of Weber and Schafheitlin whose hypergeometric form has been evaluated in [6, equations (4.4)]. When the latter result is applied to equation (3.5), we obtain, for $t > 0$,

$$\mathcal{F}\left\{ {}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2x^2\right]\right\} = 0 \qquad \left(4t^2 < \xi^2\right) \tag{3.6a}$$

and

$$\mathcal{F}\left\{ {}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2x^2\right]\right\}$$
$$= \pi^{\frac{m}{2}}\frac{\Gamma\left((b_{p+1})\right)}{\Gamma\left((a_p)\right)}\left(\frac{1}{t^m}\frac{\Gamma\left((a_p)-\frac{m}{2}\right)}{\Gamma\left((b_{p+1})-\frac{m}{2}\right)}\,{}_{p+1}F_p\left[\begin{matrix}1+\frac{m}{2}-(b_{p+1}); \\ 1+\frac{m}{2}-(a_p);\end{matrix}\;\frac{\xi^2}{4t^2}\right]\right.$$
$$+\left(\frac{2}{\xi}\right)^m\sum_{k=1}^{p}\frac{\Gamma\left(\frac{m}{2}-a_k\right)\Gamma\left((a_p)^*-a_k\right)}{\Gamma\left((b_{p+1})-a_k\right)}\left(\frac{\xi^2}{4t^2}\right)^{a_k}$$
$$\left.\times\,{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1}); \\ 1-\frac{m}{2}+a_k,\;1+a_k-(a_p)^*;\end{matrix}\;\frac{\xi^2}{4t^2}\right]\right) \qquad \left(4t^2 > \xi^2\right) \tag{3.6b}$$

where, for conciseness, $\Gamma\left((a_p)\right) := \Gamma(a_1)\cdots\Gamma(a_p)$ and

$$\Gamma\left((a_p)^*-a_k\right) := \Gamma(a_1-a_k)\cdots\Gamma(a_{k-1}-a_k)\,\Gamma(a_{k+1}-a_k)\cdots\Gamma(a_p-a_k)$$

both of which reduce to unity when $p = 0$.

Inversion of the Fourier transform given by equations (3.6) yields the following transformation formula for ${}_pF_{p+1}\left[-t^2x^2\right]$:

$${}_pF_{p+1}\left[(a_p);(b_{p+1});-t^2x^2\right] = \frac{1}{\left(2\sqrt{\pi}\right)^m}\frac{\Gamma\left((b_{p+1})\right)}{\Gamma\left((a_p)\right)}$$
$$\times\left(\frac{1}{t^m}\frac{\Gamma\left((a_p)-\frac{m}{2}\right)}{\Gamma\left((b_{p+1})-\frac{m}{2}\right)}\int_{\xi^2\leqslant 4t^2}\mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{\xi}}\,{}_{p+1}F_p\left[\begin{matrix}1+\frac{m}{2}-(b_{p+1}); \\ 1+\frac{m}{2}-(a_p);\end{matrix}\;\frac{\xi^2}{4t^2}\right]\mathrm{d}\xi\right.$$
$$+2^m\sum_{k=1}^{p}\frac{\Gamma\left(\frac{m}{2}-a_k\right)\Gamma\left((a_p)^*-a_k\right)}{\Gamma\left((b_{p+1})-a_k\right)}\left(\frac{1}{4t^2}\right)^{a_k}\int_{\xi^2\leqslant 4t^2}\mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{\xi}}\left(\xi^2\right)^{a_k-\frac{m}{2}}$$
$$\left.\times\,{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1}); \\ 1-\frac{m}{2}+a_k,1+a_k-(a_p)^*;\end{matrix}\;\frac{\xi^2}{4t^2}\right]\mathrm{d}\xi\right) \tag{3.7}$$

where $\mathrm{d}\xi = \mathrm{d}\xi_1\cdots\mathrm{d}\xi_m$ and the inequalities (1.2a) hold true.

In equation (3.7) let $\boldsymbol{x} = \boldsymbol{q}(m)$ so that $x = \left(q_1^2+q_2^2+\cdots+q_m^2\right)^{1/2}$, where $q_i \in \mathbb{Z}$, and then replace t by $x > 0$. Now, multiplying both sides of the resulting equation by

$\exp(2\pi \mathrm{i}\boldsymbol{\alpha}(m)\cdot\boldsymbol{q}(m))$ and summing over $\boldsymbol{q}(m)$, we obtain

$$\sum_{\boldsymbol{q}(m)} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{q}}\,{}_pF_{p+1}\left[(a_p);(b_{p+1});-x^2q^2\right] = \frac{1}{(2\sqrt{\pi})^m}\frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\left(\frac{\Gamma\left((a_p)-\frac{m}{2}\right)}{\Gamma\left((b_{p+1})-\frac{m}{2}\right)}\frac{1}{x^m}\right.$$
$$\times\int\limits_{\xi^2\leqslant 4x^2}\sum_{\boldsymbol{q}(m)}\mathrm{e}^{\mathrm{i}(2\pi\boldsymbol{\alpha}-\boldsymbol{\xi})\cdot\boldsymbol{q}}\,{}_{p+1}F_p\left[\begin{matrix}1+\frac{m}{2}-(b_{p+1});\\ 1+\frac{m}{2}-(a_p);\end{matrix}\ \frac{\xi^2}{4x^2}\right]\mathrm{d}\xi$$
$$+2^m\sum_{k=1}^{p}\frac{\Gamma\left(\frac{m}{2}-a_k\right)\Gamma\left((a_p)^*-a_k\right)}{\Gamma\left((b_{p+1})-a_k\right)}\left(\frac{1}{4x^2}\right)^{a_k}\int\limits_{\xi^2\leqslant 4x^2}\sum_{\boldsymbol{q}(m)}\mathrm{e}^{\mathrm{i}(2\pi\boldsymbol{\alpha}-\boldsymbol{\xi})\cdot\boldsymbol{q}}\left(\xi^2\right)^{a_k-\frac{m}{2}}$$
$$\left.\times\,{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1});\\ 1-\frac{m}{2}+a_k,1+a_k-(a_p)^*;\end{matrix}\ \frac{\xi^2}{4x^2}\right]\mathrm{d}\xi\right) \tag{3.8}$$

where $x > 0$, $\mathrm{d}\xi = \mathrm{d}\xi_1\cdots\mathrm{d}\xi_m$, and the order of summations and integrations have been interchanged in each term.

For real α_j, ξ_j and $q_j \in \mathbb{Z}$, we recall that

$$\boldsymbol{\alpha}(m) = (\alpha_1,\alpha_2,\ldots,\alpha_m)$$
$$\boldsymbol{\xi}(m) = (\xi_1,\xi_2,\ldots\xi_m)$$
$$\boldsymbol{q}(m) = (q_1,q_2,\ldots,q_m).$$

Moreover, since (for real μ)

$$\sum_{k\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}\mu k} = 2\pi\sum_{k\in\mathbb{Z}}\delta(\mu-2\pi k) \tag{3.9a}$$

where δ is the delta function (see [9, p. 189, equation (17)] or [2, equation (1.6)] where an equivalent form of equation (3.9a) is readily derived), it is easy to show that

$$\sum_{\boldsymbol{q}(m)}\mathrm{e}^{\mathrm{i}(2\pi\boldsymbol{\alpha}-\boldsymbol{\xi})\cdot\boldsymbol{q}} = (2\pi)^m\sum_{\boldsymbol{q}(m)}\prod_{j=1}^{m}\delta\left(2\pi\omega_j-\xi_j\right) \tag{3.9b}$$

where

$$\omega_j := \alpha_j - q_j. \tag{3.9c}$$

Now, replacing the $\boldsymbol{q}(m)$-summations in equation (3.8) with the right member of equation (3.9b), we have

$$\sum_{\boldsymbol{q}(m)} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{q}}\,{}_pF_{p+1}\left[(a_p);(b_{p+1});-x^2q^2\right] = \frac{\Gamma((b_{p+1}))}{\Gamma((a_p))}\left(\frac{\Gamma\left((a_p)-\frac{m}{2}\right)}{\Gamma\left((b_{p+1})-\frac{m}{2}\right)}\frac{\pi^{m/2}}{x^m}\right.$$
$$\times\sum_{\boldsymbol{q}(m)}\int\limits_{\xi^2\leqslant 4x^2}\prod_{j=1}^{m}\delta\left(2\pi\omega_j-\xi_j\right)\,{}_{p+1}F_p\left[\begin{matrix}1+\frac{m}{2}-(b_{p+1});\\ 1+\frac{m}{2}-(a_p);\end{matrix}\ \frac{\xi^2}{4x^2}\right]\mathrm{d}\xi$$
$$+\left(2\sqrt{\pi}\right)^m\sum_{k=1}^{p}\frac{\Gamma\left(\frac{m}{2}-a_k\right)\Gamma\left((a_p)^*-a_k\right)}{\Gamma\left((b_{p+1})-a_k\right)}\left(\frac{1}{4x^2}\right)^{a_k}$$
$$\times\sum_{\boldsymbol{q}(m)}\int\limits_{\xi^2\leqslant 4x^2}\prod_{j=1}^{m}\delta\left(2\pi\omega_j-\xi_j\right)\left(\xi^2\right)^{a_k-\frac{m}{2}}$$
$$\left.\times\,{}_{p+1}F_p\left[\begin{matrix}1+a_k-(b_{p+1});\\ 1-\frac{m}{2}+a_k,1+a_k-(a_p)^*;\end{matrix}\ \frac{\xi^2}{4x^2}\right]\mathrm{d}\xi\right) \tag{3.10}$$

where $\mathrm{d}\xi = \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_m$ and the order of summations and integrations have again been interchanged in both terms.

It is readily seen that, for any function $f(z)$,

$$\sum_{\boldsymbol{q}(m)} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{q}} f\left(-x^2q^2\right) = \sum_{\boldsymbol{q}(m)} \cos\left(2\pi\boldsymbol{\alpha}\cdot\boldsymbol{q}\right) f\left(-x^2q^2\right)$$

since, in the latter, $\boldsymbol{q}$ may be replaced by $-\boldsymbol{q}$. Thus, upon performing the required formal term-by-term integrations in equation (3.10) with regard to the distributional properties of the delta function, we immediately deduce for $x > 0$ that

$$\begin{aligned}\sum_{\boldsymbol{q}(m)} \cos\left(2\pi\boldsymbol{\alpha}\cdot\boldsymbol{q}\right)\, {}_pF_{p+1}\left[\left(a_p\right);\left(b_{p+1}\right);-x^2q^2\right] &= \frac{\Gamma\left(\left(b_{p+1}\right)\right)}{\Gamma\left(\left(a_p\right)\right)} \\ &\times\left(\frac{\Gamma\left(\left(a_p\right)-\frac{m}{2}\right)}{\Gamma\left(\left(b_{p+1}\right)-\frac{m}{2}\right)}\left(\frac{\sqrt{\pi}}{x}\right)^m \sum_{\boldsymbol{q}(m)}^{\omega^2\leqslant x^2/\pi^2} {}_{p+1}F_p\left[\begin{matrix}\frac{m+2}{2}-\left(b_{p+1}\right); \\ \frac{m+2}{2}-\left(a_p\right);\end{matrix}\ \frac{\pi^2\omega^2}{x^2}\right]\right. \\ &+\left(\frac{1}{\sqrt{\pi}}\right)^m \sum_{k=1}^{p} \frac{\Gamma\left(\frac{m}{2}-a_k\right)\Gamma\left(\left(a_p\right)^*-a_k\right)}{\Gamma\left(\left(b_{p+1}\right)-a_k\right)}\left(\frac{\pi^2}{x^2}\right)^{a_k} \\ &\left.\times\sum_{\boldsymbol{q}(m)}^{\omega^2\leqslant x^2/\pi^2}\left(\omega^2\right)^{a_k-\frac{m}{2}} {}_{p+1}F_p\left[\begin{matrix}1+a_k-\left(b_{p+1}\right); \\ \frac{2-m}{2}+a_k,\, 1+a_k-\left(a_p\right)^*;\end{matrix}\ \frac{\pi^2\omega^2}{x^2}\right]\right)\end{aligned} \tag{3.11}$$

where, in view of equation (3.9c), $\boldsymbol{\omega}(m) := \boldsymbol{\alpha}(m) - \boldsymbol{q}(m)$, and when $\omega^2 < x^2/\pi^2$ the conditional inequalities (1.2a) hold true. If $\omega^2 \leqslant x^2/\pi^2$, the inequalities (1.2$b$) are required in order for the generalized hypergeometric functions in equation (3.11) to converge at unit argument. Equation (3.11) has already been derived by other methods in [4, equation (1.6)], where, as previously mentioned in section 1, questions concerning the convergence of its left member are discussed in greater detail (see [4, lemma 1]).

Finally, as has also already been mentioned earlier in section 1, the equivalence of the right members of equations (1.1) and (3.11) can be seen by appealing to [6, equations (4.4) and (5.1)]. This evidently completes the derivation of equation (1.1).

4. Representation in terms of Meijer's G-function

In conclusion, we mention that, since the integral in equation (1.1) is proportional to a G-function (see [6, equation (6.1)]) when $\omega^2 < x^2/\pi^2$, we have the elegant result:

$$\begin{aligned}&\sum_{\boldsymbol{q}(m)} \cos\left(2\pi\boldsymbol{\alpha}\cdot\boldsymbol{q}\right)\, {}_pF_{p+1}\left[\left(a_p\right);\left(b_{p+1}\right);-x^2q^2\right] \\ &\quad = \frac{\Gamma\left(\left(b_{p+1}\right)\right)}{\Gamma\left(\left(a_p\right)\right)}\left(\frac{\sqrt{\pi}}{x}\right)^m \sum_{\boldsymbol{q}(m)}^{\omega^2<x^2/\pi^2} G^{p+1,0}_{p+1,p+1}\left(\frac{\pi^2\omega^2}{x^2}\,\middle|\,\begin{matrix}\left(b_{p+1}\right)-\frac{m}{2} \\ 0,\left(a_p\right)-\frac{m}{2}\end{matrix}\right)\end{aligned} \tag{4.1}$$

where $x > 0$, $\omega = |\boldsymbol{\alpha} + \boldsymbol{q}|$, $\omega^2 < x^2/\pi^2$, and the conditional inequalities (1.2a) hold true. Thus infinite m-dimensional lattice sums of generalized hypergeometric functions may be expressed essentially as a finite sum of Meijer's G-functions $G^{r,0}_{r,r}(z)$, where $0 < z < 1$.

Mathai [3] has studied the latter function in [3, lemma 1, and theorems 1 and 2], where explicit series representations and other results are given for it.

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